Analysis of microswimmers: from one to many

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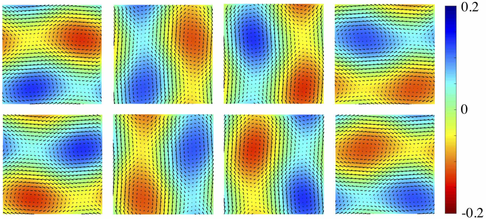
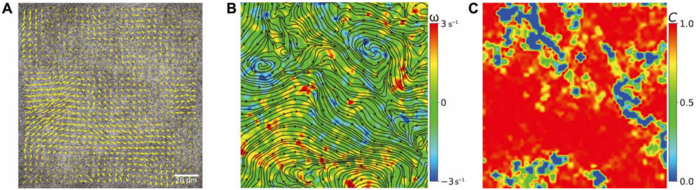
Symposium on computational math for engineering and sciences
Penn State

November 14, 2023



[Ovation Fertility 2017, Kantsler 2017]



[Peng–Liu–Cheng 2021]

I. Single swimmer

Undulatory swimming via resistive force theory

Immersed, inextensible elastic filament $\boldsymbol{X} : [0, L] \times [0, T] \to \mathbb{R}^3$:

1. Resistive force theory:

$$\frac{\partial \boldsymbol{X}}{\partial t}(s,t) = -c_{\rm h}(\mathbf{I} + \boldsymbol{X}_s\boldsymbol{X}_s^{\rm T})\mathbf{f}_{\rm h}(s,t)$$

$$c_{\rm h} = \frac{|\log(\epsilon/L)|}{4\pi}$$

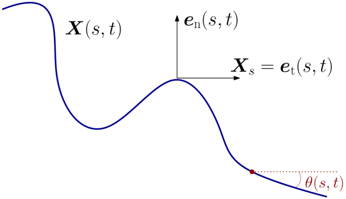
2. Euler-Bernoulli beam theory:

$$\mathbf{f}_{\rm h}(s,t) = \big(E(\boldsymbol{X}_{sss} - (\kappa_0)_s\boldsymbol{e}_{\rm n}) - \tau\boldsymbol{X}_s\big)_s\,, \quad |\boldsymbol{X}_s|^2 = 1$$

Here $\kappa_0(s,t)$: simple representation of internal mechanics (see [Fauci–Peskin 1988, Camalet–Jülicher 2000, Thomases–Guy 2017])

3. Force-free and torque-free:

$$\int_0^L \mathbf{f}_{\rm h}(s,t) = 0\,, \qquad \int_0^L \boldsymbol{X}(s,t) \times \mathbf{f}_{\rm h}(s,t) = 0$$



Together (rescaling time as $\frac{Ec_{\rm h}}{L^4}t$):

$$\frac{\partial \boldsymbol{X}}{\partial t}(s,t) = -\left(\mathbf{I} + \boldsymbol{X}_s\boldsymbol{X}_s^{\rm T}\right)\left(\boldsymbol{X}_{sss} - (\kappa_0)_s\boldsymbol{e}_{\rm n} - \tau(s,t)\boldsymbol{X}_s\right)_s$$

$$|\boldsymbol{X}_s|^2 = 1$$

$$\left(\boldsymbol{X}_{ss} - \kappa_0\boldsymbol{e}_{\rm n}\right)\Big|_{s=0,1} = 0\,, \quad \left(\boldsymbol{X}_{sss} - (\kappa_0)_s\boldsymbol{e}_{\rm n} - \tau\boldsymbol{X}_s\right)\Big|_{s=0,1} = 0$$

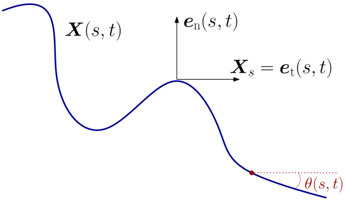
When κ_0 is time-independent, evolution seeks to minimize bending energy:

$$\frac{1}{2}\frac{d}{dt}\int_0^1 (\kappa-\kappa_0)^2\, ds = -\int_0^1 \left((\kappa-\kappa_0)_{ss} - \kappa^3 - \tau\kappa\right)^2 ds$$
$$- 2\int_0^1 \left(3\kappa\kappa_s - \kappa(\kappa_0)_s + \tau_s\right)^2 ds < 0$$

(where $\kappa = \boldsymbol{X}_{ss}\cdot\boldsymbol{e}_{\rm n}$)

Goal: Given a *time-dependent* preferred curvature κ_0, study the PDE evolution, particularly the inextensibility constraint. Prove conditions on κ_0 allowing the filament to swim, and test predictions numerically.

Tangent angle formulation:



$$\dot{\theta} = -\theta_{ssss} + (\kappa_0)_{sss} + \mathcal{N}[\theta_s, \kappa_0]$$
$$\tau_{ss} = \tfrac{1}{2}\theta_s^2 \tau + \mathcal{T}[\theta_s, \kappa_0]$$

Curvature formulation: ($\kappa = \theta_s$, $\overline{\kappa} = \kappa - \kappa_0$, $\overline{\tau} = \tau + \kappa_0^2$):

$$\dot{\overline{\kappa}} = -\overline{\kappa}_{ssss} - \dot{\kappa}_0 + \big(\mathcal{N}[\overline{\kappa}, \kappa_0]\big)_s$$
$$\overline{\tau}_{ss} = \tfrac{1}{2}(\overline{\kappa} + \kappa_0)^2 \overline{\tau} + \mathcal{T}[\overline{\kappa}, \kappa_0]$$
$$\overline{\kappa}\big|_{s=0,1} = \overline{\kappa}_s\big|_{s=0,1} = \overline{\tau}\big|_{s=0,1} = 0$$

$$\mathcal{N}[\overline{\kappa}, \kappa_0] := 9\overline{\kappa}(\overline{\kappa} + 2\kappa_0)\overline{\kappa}_s + 8\kappa_0^2\overline{\kappa}_s + 7\overline{\kappa}^2(\kappa_0)_s + 8\overline{\kappa}\kappa_0(\kappa_0)_s + 3\overline{\tau}_s(\overline{\kappa} + \kappa_0) + \overline{\tau}(\overline{\kappa} + \kappa_0)_s$$
$$2\mathcal{T}[\overline{\kappa}, \kappa_0] := \overline{\kappa}(\overline{\kappa} + \kappa_0)^2(\overline{\kappa} + 2\kappa_0) + (\overline{\kappa} + \kappa_0)_s\overline{\kappa}_s - 2\big(\overline{\kappa}(\overline{\kappa} + 2\kappa_0)\big)_{ss} - 3\big(\overline{\kappa}_s(\overline{\kappa} + \kappa_0)\big)_s$$

Consider: $\partial_{ssss}\psi, \quad \psi|_{s=0,1} = \psi_s|_{s=0,1} = 0$

Eigenvalues: $\lambda_k = \xi_k^4$ where $\cos(\xi_k)\cosh(\xi_k) = 1\,,\ \xi_0 = 0$

Eigenfunctions: $\psi_k = A_k\left(\cos(\xi_k s) - \cosh(\xi_k s)\right) + B_k\left(\sin(\xi_k s) - \sinh(\xi_k s)\right)$

Theorem (Well-posedness [Mori–O. Nonlin. 2023])

Given a sufficiently small $\kappa_0 \in C^1([0,T]; H^1)$,

1. *There is a time $T^*(\overline{\kappa}^{\rm in})$ s.t. a unique solution $\overline{\kappa}$ exists up to time T^*.*
2. *If $\|\overline{\kappa}^{\rm in}\|_{L^2}$ is sufficiently small, for any $T > 0$, a unique solution $\overline{\kappa}$ exists and satisfies*

$$\sup_{t\in[0,T]}\left(\|\overline{\kappa}\|_{L^2} + \min\{t^{1/4},1\}\,\|\overline{\kappa}\|_{\dot{H}^1}\right) \le c\left(\|\kappa^{\rm in}\|_{L^2} + \|\kappa_0\|_{H^1} + \|\dot{\kappa}_0\|_{L^2}\right).$$

If $\kappa_0 \equiv 0$,

$$\|\kappa\|_{L^2} + \min\{t^{1/4},1\}\,\|\kappa\|_{\dot{H}^1} \le c\, e^{-t\lambda_1}\|\kappa^{\rm in}\|_{L^2}\,.$$

Theorem (Swimming [Mori–O. Nonlin. 2023])

Suppose that $\kappa_0(s,t) \in C^1([0,T];H^3)$ is T-periodic in time and sufficiently small. The filament swims with speed

$$U(t) = -\int_0^1 (\kappa_0)_s(\kappa - \kappa_0)\,ds + O(\|\kappa_0\|^3)\,.$$

Writing $\kappa_0(s,t) = \sum_{m,k=1}^{\infty} \big(a_{m,k}\cos(\omega m t) - b_{m,k}\sin(\omega m t)\big)\psi_k(s)$, $\omega = \frac{2\pi}{T}$:

$$\frac{1}{T}\int_0^T U\,dt = \frac{1}{2}\sum_{m,k,\ell=1}^{\infty} \frac{\omega^2 m^2}{\omega^2 m^2 + \lambda_k^2}\bigg(\frac{\lambda_k}{\omega m}(a_{m,k}b_{m,\ell} - b_{m,k}a_{m,\ell}) + a_{m,k}a_{m,\ell} + b_{m,k}b_{m,\ell}\bigg)\int_0^1 \psi_k(\psi_\ell)_s\,ds + O(\|\kappa_0\|^3)\,.$$

- Swimming speed scales like $\|\kappa_0\|^2$ ([Taylor 1951]: square of amplitude)
- Valid at finite bending stiffness:
  $t = Et' \Longrightarrow U(t') = -E\int_0^1 (\kappa_0)_s(\kappa - \kappa_0)\,ds$

Consider forcing at lowest nonzero temporal frequency only:

$$\kappa_0(s,t) = F_1(s)\cos(\omega\, t) + F_2(s)\sin(\omega\, t)$$

Then (leading order) swimming speed is

$$\frac{1}{2}\sum_{k,\ell=1}^{\infty} \frac{\omega^2}{\omega^2+\lambda_k^2}\left(\frac{\lambda_k}{\omega}\left(a_k b_\ell - b_k a_\ell\right) + a_k a_\ell + b_k b_\ell\right)\int_0^1 \psi_k (\psi_\ell)_s \, ds$$

"Scallop Theorem" for elastic swimmers: (Conditions on κ_0, not actual motion)

1. If F_1 and F_2 are both even or both odd about $s = \frac{1}{2}$, the integral $\int_0^1 \psi_k(\psi_\ell)_s \, ds$ vanishes and the filament does not swim.
2. If $F_1 = 0$, $F_2 = 0$, or $F_1 = \pm F_2$, then the first term vanishes. The filament may still swim, but its displacement will be very small, due to the size of λ_k.

Note that a traveling wave κ_0 avoids both (1) and (2).

What is the optimal κ_0 for swimming?

Given $\kappa_0 = \sum_{k=1}^{\infty} \big(a_k \cos(\omega t) - b_k \sin(\omega t)\big)\psi_k(s)$, consider average work:

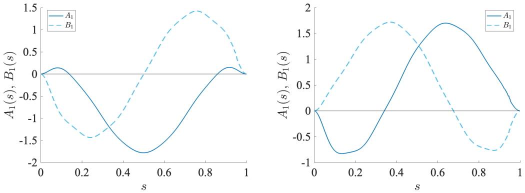
$$\frac{1}{T}\int_0^T W\,dt := \frac{1}{T}\int_0^T \int_0^1 \dot{\kappa}_0(\kappa - \kappa_0)\,ds\,dt \approx \sum_{k=1}^{\infty} \frac{\lambda_k}{2}\frac{\omega^2}{\omega^2+\lambda_k^2}(a_{m,k}^2 + b_{m,k}^2)$$

Define $U_{k_{\max}}$, $W_{k_{\max}}$ to be swimming speed and work using first $k_{\max}$ modes. Solve:

$$\min_{a_k, b_k} \quad U_{k_{\max}}$$

$$\text{subject to} \quad W_{k_{\max}} = 1\,, \quad \sum_k a_k^2 = \sum_k b_k^2 = 1\,.$$

Solution: (Same up to time translation)



Will compare to classic traveling wave $F_1 = \sin(\omega s)$ and $F_2 = \cos(\omega s)$

Inspired by [Moreau et al. 2018, Maxian et al. 2021], rather than solve BVP for τ, *enforce inextensibility directly* in parameterization:

$$\boldsymbol{X}(s,t) = \boldsymbol{X}_0(t) + \int_0^s \boldsymbol{X}_s(s',t)ds', \quad \boldsymbol{X}_s = \boldsymbol{e}_\mathrm{t} = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}.$$

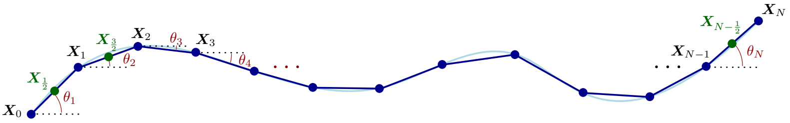
Recast evolution:

$$\boldsymbol{e}_\mathrm{n}(s,t) \cdot \int_0^s \boldsymbol{f}(s',t)\,ds' = -\theta_{ss} + (\kappa_0)_s\,,$$

where $\boldsymbol{f}(s,t) = (\mathbf{I} + \boldsymbol{e}_\mathrm{t}\boldsymbol{e}_\mathrm{t}^\mathrm{T})^{-1}\frac{\partial \boldsymbol{X}}{\partial t} = (\mathbf{I} - \frac{1}{2}\boldsymbol{e}_\mathrm{t}\boldsymbol{e}_\mathrm{t}^\mathrm{T})\big(\dot{\boldsymbol{X}}_0 + \int_0^s \dot{\boldsymbol{e}}_\mathrm{t}(s')\,ds'\big)$.

Accompany with $\int_0^1 \boldsymbol{f}(s,t)\,ds = 0$ to enforce $(-\theta_{ss} + (\kappa_0)_s)\big|_{s=1} = 0$.

Discretize fiber into N segments and enforce at midpoints:



Have $N+2$ equations for $N+2$ unknowns: $\boldsymbol{X}_0$ and θ_j, $j = 1, \ldots, N$.

Numerical results

Non-swimmer

Bad swimmer

Classic traveling wave

Optimum 1

Optimum 2

- *Resistive force theory dynamics as limit of PDE in the bulk*
  We have achieved this for nonlocal slender body theory in the static setting in [Mori-O.-Spirn CPAM 2020, Mori-O.-Spirn ARMA 2020, Mori-O. SAPM 2021, etc.]
  Progress in dynamic setting in [O. 2023]

- *What is the best way to implement an inextensibility constraint?*
  Quantify the differences between projection methods versus direct discretization of curve evolution via numerical analysis

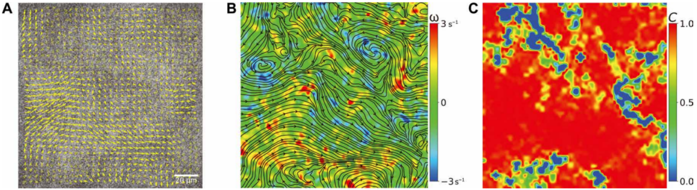
- *Swimming questions*
  PDE-constrained optimization of preferred curvature for swimming; resistive force theories in viscoelastic media [Ohm 2022]; preferred curvature as limit of micromechanical description of filament motion

II. Many swimmers

Collective behavior via kinetic theory

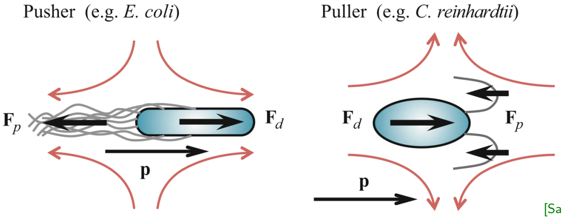
Active suspensions and bacterial turbulence

[Kantsler 2017]



[Peng–Liu–Cheng 2021]

Each swimmer is approximated by a *force dipole*:



[Saintillan-Shelley 2013]

The flow field around a single swimmer is approximately

$$-\mu\Delta \boldsymbol{u} + \nabla q = \pm \frac{F\ell}{2} \underbrace{(\boldsymbol{p}\cdot\nabla_x)\delta(\boldsymbol{x})\boldsymbol{p}}_{=\mathrm{div}_x(\boldsymbol{p}\otimes\boldsymbol{p}\,\delta(\boldsymbol{x}))}, \quad \mathrm{div}\,\boldsymbol{u} = 0\,.$$

Rod-like particles swim with speed V_0 and are transported:

$$\dot{\boldsymbol{x}} = V_0\boldsymbol{p} + \boldsymbol{u}\,, \quad \dot{\boldsymbol{p}} = (\mathbf{I} - \boldsymbol{p}\otimes\boldsymbol{p})(\nabla \boldsymbol{u}\boldsymbol{p})\,.$$

$$\partial_t \psi + \nabla_x \cdot (\dot{\boldsymbol{x}}\psi) + \nabla_p \cdot (\dot{\boldsymbol{p}}\psi) = \kappa \Delta_x \psi + \nu \Delta_p \psi$$

$$\dot{\boldsymbol{x}} = \boldsymbol{p} + \boldsymbol{u}$$

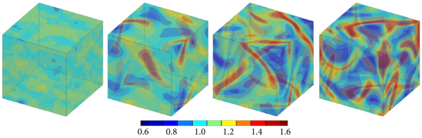
$$\dot{\boldsymbol{p}} = (\mathbf{I} - \boldsymbol{p} \otimes \boldsymbol{p})(\nabla \boldsymbol{u}\, \boldsymbol{p})$$

$$-\Delta \boldsymbol{u} + \nabla q = \nabla_x \cdot \boldsymbol{\Sigma}, \quad \operatorname{div} \boldsymbol{u} = 0$$

$$\boldsymbol{\Sigma}(\boldsymbol{x}, t) = \iota \int_{S^{d-1}} \boldsymbol{p} \otimes \boldsymbol{p}\, \psi(\boldsymbol{x}, \boldsymbol{p}, t)\, d\boldsymbol{p}, \quad \iota \in \{\pm\}$$

[Saintillan–Shelley 2008]

$\psi(\boldsymbol{x}, \boldsymbol{p}, t)$: # of swimmers at $\boldsymbol{x} \in \mathbb{T}^d$ with orientation $\boldsymbol{p} \in S^{d-1}$
$\boldsymbol{u}(\boldsymbol{x}, t)$, $q(\boldsymbol{x}, t)$: fluid velocity & pressure
$\boldsymbol{\Sigma}(\boldsymbol{x}, t)$: signed active stress ($+$ pullers, $-$ pushers)
ν, κ: (nondimensional) diffusion coefficients
$\overline{\psi}$: (nondimensional) number density of swimmers $\left(\overline{\psi} = \frac{FLn}{2\pi\mu V_0}\right)$



[Saintillan–Shelley 2013]

What do we know? [Hohenegger–Shelley 2010]

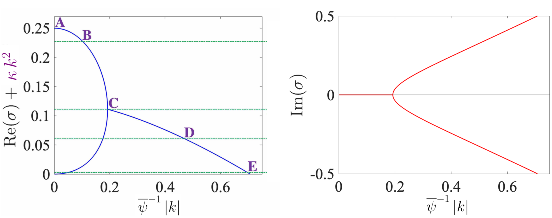
Linear stability analysis shows unstable eigenvalue(s) for large $\overline{\psi}$ in pusher suspensions ($\iota = -$). In 2D, consider

$$\psi = h(\boldsymbol{k}, \boldsymbol{p})\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}+\sigma t}, \ \sigma \in \mathbb{C}, \quad \boldsymbol{p} = \cos\theta \boldsymbol{e}_x + \sin\theta \boldsymbol{e}_y$$

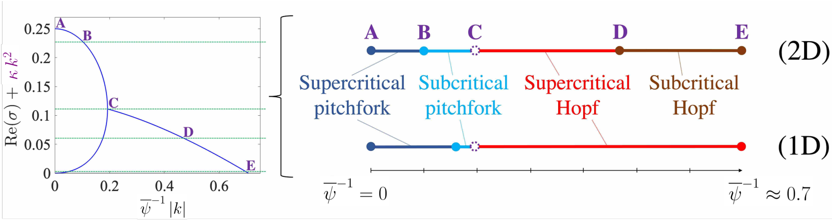
If $\nu = 0$, growth rate σ satisfies (note: $\beta \sim 1/\overline{\psi}$)

$$\int_0^{2\pi} \frac{\cos^2\theta \sin^2\theta}{\sigma + \kappa\, k^2 + ik\beta\cos\theta}\, d\theta = \pi$$

Solve for σ numerically:



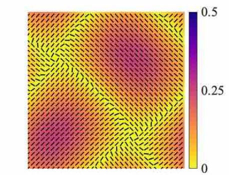
[Ohm–Shelley JFM 2022]



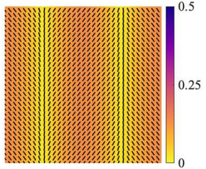
Pusher instability: 2D patterns

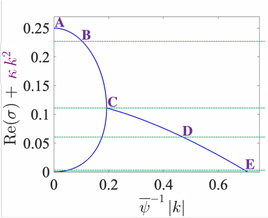
[Ohm–Shelley JFM 2022]

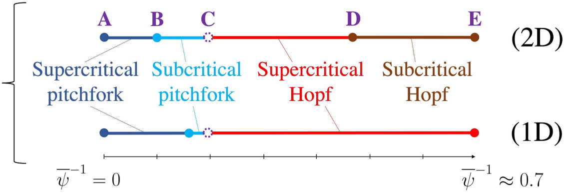
(2D):



(1D):







What about stability?

Without swimming, the uniform isotropic pusher equilibrium $\psi = \overline{\psi}$ is *always* unstable (for $\nu, \kappa \ll 1$).

How does swimming stabilize the uniform isotropic steady state $\psi \equiv \overline{\psi}$?

This kinetic model shares similarities with more standard kinetic theories for many-particle systems (Vlasov–Poisson, Boltzmann, etc.)

Can adapt the tools and language for studying stability in these more standard settings to answer this question.

Back to the kinetic model [Saintillan–Shelley 2008]

$$\partial_t \psi + \boldsymbol{p} \cdot \nabla_x \psi + \boldsymbol{u} \cdot \nabla_x \psi + \nabla_p \cdot [(\mathbf{I} - \boldsymbol{p} \otimes \boldsymbol{p})(\nabla \boldsymbol{u} \boldsymbol{p})\psi] = \nu \Delta_p \psi + \kappa \Delta_x \psi$$

$$-\Delta \boldsymbol{u} + \nabla q = \nabla_x \cdot \boldsymbol{\Sigma}, \quad \operatorname{div} \boldsymbol{u} = 0$$

$$\boldsymbol{\Sigma}(\boldsymbol{x}, t) = \iota \int_{S^{d-1}} \boldsymbol{p} \otimes \boldsymbol{p}\, \psi(\boldsymbol{x}, \boldsymbol{p}, t)\, d\boldsymbol{p}, \quad \iota \in \{\pm\}$$

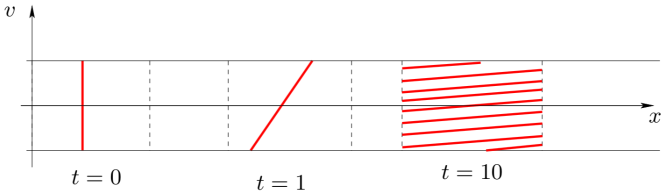
We quantify three stabilizing effects of swimming:

1. *Landau damping:* Decay of solutions to the linearized "inviscid" ($\nu = \kappa = 0$) equations on $\mathbb{T}^d$, $d = 2, 3$
2. *Taylor dispersion:* On $\mathbb{R}^3$, nonlinear stability of $\overline{\psi} = 0$ due to dispersive effect of $\boldsymbol{p} \cdot \nabla_x - \nu \Delta_p$
3. *Enhanced dissipation:* Nonlinear stability of $\psi = \overline{\psi}$ (small) on $\mathbb{T}^d$ with convergence to $\langle \psi \rangle(\boldsymbol{p}, t) := \int \psi(\boldsymbol{x}, \boldsymbol{p}, t)\, d\boldsymbol{x}$ in time $O(\nu^{-\frac{1}{2}})$

In (linearized) inviscid setting ($\nu = \kappa = 0$), stability for $\psi = \overline{\psi}$ on $\mathbb{T}^d$ is due to orientation mixing from swimming

Familiar example: phase mixing in transport equation, $f = f(x, v, t)$:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = 0, \quad x, v \in \mathbb{T}^d \times \mathbb{R}^d$$



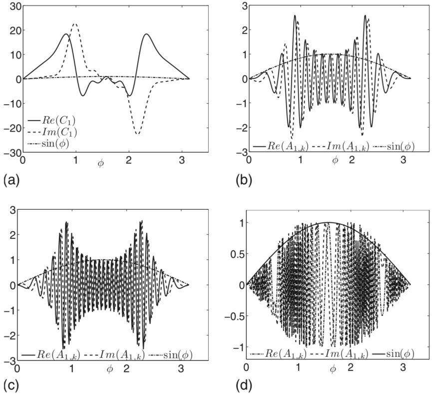
[Villani 2010]

On Fourier side: $\frac{\partial \widehat{f}}{\partial t} - k \cdot \nabla_\eta \widehat{f} = 0;$ i.e. $\widehat{f}(t, k, \eta) = \widehat{f}^{\rm in}(k, \eta + kt)$

If $f^{\rm in} \in W_v^{\ell,1}$: $|\widehat{f}(t, k, \eta)| = |\widehat{f}^{\rm in}(k, \eta + kt)| \le C|\eta + kt|^{-\ell} \to 0, \quad k \neq 0$

Orientation mixing

Swimming has similar effect:



[Hohenegger-Shelley 2010]

- For each $\boldsymbol{k} \neq 0$, solution is transferred to higher modes in $\boldsymbol{p}$ over time (weak convergence to zero via oscillations in $\boldsymbol{p}$)
- Concentration $c(\boldsymbol{x}, t) = \int_{S^{d-1}} \psi(\boldsymbol{x}, \boldsymbol{p}, t)\, d\boldsymbol{p}$ converges to mean; $\boldsymbol{k} = 0$ unchanged

Linearized equation in inviscid setting ($\nu = \kappa = 0$):

$$\partial_t f + \boldsymbol{p} \cdot \nabla_x f - d\overline{\psi} \nabla \boldsymbol{u} : \boldsymbol{p} \otimes \boldsymbol{p} = 0$$

Kinetic free swimming: $\overline{\psi} = 0$. Write $f = he^{i\boldsymbol{k} \cdot \boldsymbol{x}}$: (take $\boldsymbol{k} = k\boldsymbol{e}_1$, $t \mapsto t/k$)

$$\partial_t h + ip_1 h = 0 \,, \quad h(\cdot, 0) = h^{\text{in}} \,.$$

Writing $p_1 = \cos\theta$, oscillations grow over time except where $\partial_\theta p_1 = 0$, which limits decay:

$$\|h\|_{H^{-(d-1)}} \lesssim \langle t \rangle^{-\frac{d-1}{2}} \|h^{\text{in}}\|_{H^{d-1}}$$

Compare to transport equation (Vlasov-Poisson): $v \cdot \nabla_x$ gives exponential decay to mean-in-x if f^{in} is analytic-in-v.

Incorporate nonlocal term $\overline{\psi} > 0$. Obtain Volterra equation for $\widehat{\nabla \boldsymbol{u}}$: $(\boldsymbol{k} = k\boldsymbol{e}_1)$

$$\widehat{\nabla \boldsymbol{u}}[h] = \widehat{\nabla \boldsymbol{u}}[e^{-ip_1 t} h^{\mathrm{in}}] - \iota d\overline{\psi} \int_0^t K(t-s)\, \widehat{\nabla \boldsymbol{u}}[h](s)\, ds\,.$$

Taking Fourier-Laplace transform L, may (formally) solve:

$$L\widehat{\nabla \boldsymbol{u}}[h] = (I + \iota d\overline{\psi} LK)^{-1} L\widehat{\nabla \boldsymbol{u}}[e^{-ip_1 t} h^{\mathrm{in}}]\,.$$

Theorem (Linear Landau damping, Albritton-Ohm SIMA 2023)

Let $f^{\mathrm{in}} \in L^2_x H^{d+1}_p(\mathbb{T}^d \times S^{d-1})$. Suppose $\iota = +$ or $\overline{\psi} < \overline{\psi}^*$. Then $(\langle t \rangle = \sqrt{1+t^2})$

$$\int \|\nabla \boldsymbol{u}(\cdot, t)\|^2_{L^2_x} \langle t \rangle^{d-\varepsilon}\, dt \lesssim_{\overline{\psi},\varepsilon} \|f^{\mathrm{in}}\|^2_{L^2_x H^{d+1}_p}\,.$$

(Sharpened to L^∞-in-time bound in [Coti Zelati–Dietert–Gerard Varet 2023])

Stability threshold $\overline{\psi}^*$ for pushers $(\iota = -)$ arises as a *Penrose condition* which is equivalent to no solution with $\mathrm{Re}(\lambda) \geq 0$ to

$$\iota d\overline{\psi} \int_{S^{d-1}} \frac{p_1^2 p_j^2}{\lambda + ip_1}\, d\boldsymbol{p} = 1$$

(i.e. the linearized operator has no unstable/marginally stable eigenvalue)

Generalized Taylor dispersion

Now consider $0 < \nu, \kappa \ll 1$ and linearized PDE with $\overline{\psi} = 0$:

$$\partial_t f + \boldsymbol{p} \cdot \nabla_x f = \nu \Delta_p f + \kappa \Delta_x f \, .$$

For swimmers with speed U_0, predict effective $\boldsymbol{x}$-diffusion

$$\left(\kappa + \frac{U_0^2}{2d\nu} \right) \Delta_x$$

(see [Saintillan-Shelley 2015, Lauga 2020])

Generalized Taylor dispersion: inverse dependence of effective viscosity on ν

[Taylor 1954, Frankel 1989]

Let $f = he^{i\boldsymbol{k}\cdot\boldsymbol{x}}$, $k = |\boldsymbol{k}|$ (can take $\kappa = 0$)

Lemma (Linear Taylor dispersion, Albritton–Ohm SIMA 2023)

$$\|h(\cdot, t)\|_{L^2_p} \lesssim e^{-c_0 \mu_{\nu,k} t} \|h^{\mathrm{in}}\|_{L^2_p} \, , \quad \textit{where } \mu_{\nu,k} = \begin{cases} \frac{k^2}{\nu}, & k \le \nu \\ \nu, & k \ge \nu \end{cases}$$

Corollary:

- Nonlinear stability of $\psi = 0$ on $\mathbb{R}^3$
- Nonlinear stability of *any* $\psi = \overline{\psi}$ for pullers on $\mathbb{T}^d$.

Can do better than ν^{-1} timescale for small $\overline{\psi} \ll \nu^{-\frac{1}{2}}$

([Coti Zelati–Dietert–Gerard Varet 2023]: linear enhancement for all $\overline{\psi}$ satisfying Penrose)

Due to the *hypocoercive* effect of $\boldsymbol{p} \cdot \nabla_x - \nu \Delta_p$,
$\psi(\boldsymbol{x}, \boldsymbol{p}, t) \to \langle \psi \rangle(\boldsymbol{p}, t) := \int \psi(\boldsymbol{x}, \boldsymbol{p}, t)\, d\boldsymbol{x}$ in *enhancement time* $O(\nu^{-\frac{1}{2}-})$.

The $\boldsymbol{x}$-averages $\langle \psi \rangle$ converge to $\overline{\psi}$ in *diffusive time* $O(\nu^{-1})$.

This effect is better visualized for shear flows:

Nonlinear evolution of $f = \psi - \overline{\psi}$:

$$\partial_t f + \boldsymbol{p} \cdot \nabla_x f - \iota d\overline{\psi} \nabla_x \boldsymbol{u} : \boldsymbol{p} \otimes \boldsymbol{p} - \nu \Delta_p f - \kappa \Delta_x f = -\boldsymbol{u} \cdot \nabla_x f - \mathrm{div}_p [(\mathbf{I} - \boldsymbol{p} \otimes \boldsymbol{p})(\nabla \boldsymbol{u}[f] \boldsymbol{p}) f] \,.$$

Only nonzero spatial modes enhance. Consider f_0 and $f_{\neq}$ separately:

Theorem (Nonlinear enhanced dissipation, Albritton–Ohm SIMA 2023)

Suppose $f^{\mathrm{in}} \in H^1_x L^2_p(\mathbb{T}^d \times S^{d-1})$ *and* $\overline{\psi} \ll \nu^{1/2+}$*. If*

$$\varepsilon := \|f_{\neq}^{\mathrm{in}}\|_{H^2_x L^2_p} \leq \varepsilon_0 \quad \text{and} \quad \|f_0^{\mathrm{in}}\|_{L^2_p} \leq \varepsilon_0, \qquad 0 < \varepsilon_0 \ll \min(\kappa^{3/4+}, \nu^{3/4+}),$$

then the nonzero modes of f *satisfy the enhanced decay rate*

$$\|f_{\neq}(\cdot, t)\|_{H^2_x L^2_p} \lesssim e^{-c_{\neq} \lambda_\nu t} \varepsilon \,, \quad \lambda_\nu = \frac{\nu^{1/2}}{1 + |\log \nu|}$$

Furthermore, the zero mode satisfies

$$\|f_0\|_{L^2_p} \lesssim e^{-c_0 \nu t} \left(\|f_0^{\mathrm{in}}\|_{L^2_p} + \nu^{-1} \varepsilon^2 \right) .$$

What's next?

- *Complete near-equilibrium understanding of the model*
  Precise asymptotics of the generalized Taylor dispersion & stability of $\psi = 0$ on $\mathbb{R}^2$ (see [Beck–Wayne–Chaudhary 2017]). Dispersion of swimmers in inviscid setting; nonlinear Landau damping.

- *Boundary effects*
  Develop PDE theory of swimmers in bounded domains – particularly in the absence of translational diffusion $\kappa \to 0$. Identify steady states and their stability.

- *Far-from-equilibrium dynamics*
  Have global well-posedness for $\kappa > 0$. Estimate Hausdorff dimension of global attractor (# of degrees of freedom of turbulent bacterial suspension)

- *Mixing and transport in more complicated flows*
  Quantify effects of swimming in shear flows and cellular flows (see [Ran, et al. 2021]).

Thanks for listening!

Questions?